

Mathematical Reasoning in Zen Models

Technical Report v2025.08

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August 2025

Abstract

We present a comprehensive study of mathematical reasoning capabilities in Zen models, covering training data construction, process reward models (PRMs) for step-level supervision, and a symbolic-neural hybrid verification system. Zen achieves 73.2% on the MATH benchmark, 96.4% on GSM8K, 42.1% on AIME 2024, 78.4% on AMC 2023, and 31.2% on FrontierMath. Our step-level PRM improves MATH accuracy by 8.1 points over outcome-only reward models. The hybrid verification system using SymPy for algebraic verification and neural methods for proof checking enables reliable answer verification across problem classes. We describe data construction, training protocol, and detailed performance analysis by mathematical domain.

1 Introduction

Mathematical reasoning requires multi-step inference, precise computation, and systematic verification – capabilities that push language models to their limits. Unlike open-ended generation tasks, mathematics has ground truth: answers are either correct or incorrect, enabling automated evaluation and scalable training signal generation.

The Zen mathematical reasoning stack consists of three components:

1. **Mathematical pretraining data:** 120B tokens of mathematical text spanning textbooks, papers, and competition problems.
2. **Process Reward Model (PRM):** A model that evaluates solution quality step-by-step rather than only at the final answer.
3. **Hybrid verification:** Symbolic algebra (SymPy) + neural verification for reliable answer checking.

2 Mathematical Pretraining Data

2.1 Corpus Composition

Source	Tokens	Quality
arXiv math papers	38B	High
Math textbooks (OCR)	22B	High
Competition problems + solutions	4B	Very high
Stack Exchange (math)	18B	Medium
Synthetic generated problems	24B	High
Wikipedia math articles	8B	High
Code (mathematical libraries)	6B	Medium
Total	120B	–

Table 1: Mathematical pretraining corpus.

2.2 Synthetic Problem Generation

A core challenge is data scarcity at the frontier: competition-level problems are rare and quickly memorized. We generate synthetic problems via three strategies:

2.2.1 Template Perturbation

Existing problems are modified by changing numerical constants, function types, or geometric configurations while preserving problem structure. For each seed problem, we generate 10–20 variants with verified solutions.

2.2.2 Forward-Generation

We sample mathematical objects (functions, sequences, geometric configurations) and generate problems whose answer is the sampled object. For example: sample a polynomial $p(x)$, generate a word problem whose solution is $p(x)$.

2.2.3 Backward-Chaining

Starting from a target theorem or formula, generate a minimal problem that requires the target as a solution step. This biases the distribution toward problems that exercise specific mathematical tools.

Strategy	Problems Generated	Verified Correct
Template perturbation	4.2M	97.3%
Forward-generation	2.8M	94.1%
Backward-chaining	1.6M	91.8%
Total	8.6M	94.9%

Table 2: Synthetic problem generation statistics.

3 Process Reward Model

3.1 Motivation

Outcome reward models (ORMs) assign a single score to the complete solution. This signal is sparse for long multi-step problems and provides no information about which steps are correct. Process reward models (PRMs) evaluate each reasoning step, providing dense supervision:

$$r_{\text{PRM}}(s_1, s_2, \dots, s_T) = \prod_{t=1}^T p(\text{step}_t \text{ correct} \mid s_1, \dots, s_t) \quad (1)$$

3.2 PRM Training Data: Process Supervision

We collect process supervision data via two methods:

Human annotation: 12K solutions to MATH and AMC problems annotated step-by-step (correct/incorrect/uncertain) by mathematically trained annotators. Inter-annotator agreement: 91.3% on step-level labels.

Monte Carlo sampling: For a solution prefix $(s_1, \dots, s_t)$, we estimate step correctness by sampling $K = 128$ completions and computing the fraction that reach the correct final answer:

$$p(\text{step}_t \text{ correct}) \approx \frac{1}{K} \sum_{k=1}^K \mathbf{1}[\text{completion}_k \text{ correct}] \quad (2)$$

Algorithm 1 Monte Carlo PRM Data Generation

Require: Problem set $\mathcal{P}$, policy π , completions per step $K = 128$

Ensure: Step-label dataset $\mathcal{D}_{\text{prm}}$

```
1: for each problem  $p \in \mathcal{P}$  do
2:    $S \leftarrow \pi.\text{solve}(p)$  {Generate full solution  $S = (s_1, \dots, s_T)$ }
3:   for step  $t = 1 \dots T$  do
4:     completions  $\leftarrow [\pi(p, s_{1:t}) \text{ for } k = 1 \dots K]$ 
5:      $\hat{q}_t \leftarrow \frac{1}{K} \sum_k \mathbf{1}[\text{correct}(\text{completion}_k)]$ 
6:      $\mathcal{D}_{\text{prm}} \leftarrow \mathcal{D}_{\text{prm}} \cup \{(p, s_{1:t}, \hat{q}_t)\}$ 
7:   end for
8: end for
9: return  $\mathcal{D}_{\text{prm}}$ 
```

3.3 PRM Architecture and Training

The PRM is a Zen-7B model with a classification head on each reasoning step token. Training uses a binary cross-entropy loss with soft labels from MC sampling:

$$\mathcal{L}_{\text{PRM}} = - \sum_t [\hat{q}_t \log p_\phi(c_t) + (1 - \hat{q}_t) \log(1 - p_\phi(c_t))] \quad (3)$$

We train on 800K step-labeled solutions (600K synthetic + 200K human-annotated).

3.4 PRM-Guided Search

At inference, we use beam search guided by the PRM:

$$\text{score}(s_{1:T}) = \min_t p_\phi(c_t \mid s_{1:t}) \quad (\text{pessimistic PRM}) \quad (4)$$

The pessimistic aggregation (minimum over steps) prioritizes solutions with no weak steps over solutions with some strong steps and some weak ones.

4 Hybrid Verification

4.1 Symbolic Verification (SymPy)

For algebraic expressions, equations, and numerical answers, we use SymPy to verify equivalence:

Algorithm 2 Symbolic Verification

Require: Predicted answer $\hat{a}$, ground truth a^*

Ensure: Boolean: correct or not

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1: Parse  $\hat{a}$  and  $a^*$  into SymPy expressions
2: if simplify( $\hat{a} - a^*$ ) = 0 then
3:   return True
4: else if simplify( $\hat{a} - a^*$ )  $\neq$  0 then
5:   return False
6: else
7:
8:   return NeuralVerify( $\hat{a}, a^*$ )
9: end if
```

Symbolic verification handles polynomial equations, trigonometric identities, and rational expressions. It fails on open-form answers (proofs, geometric arguments, combinatorial constructions), which are routed to neural verification.

4.2 Neural Verification

For non-algebraic answers, a Zen-32B model trained on 50K verified proof/answer pairs assesses correctness. The verifier is given both the predicted answer and the ground truth and produces a calibrated correctness probability.

5 Experiments

5.1 Benchmark Results

Model	MATH	GSM8K	AIME 2024	AMC 2023	FrontierMath
Zen-7B	71.4	95.8	38.3	74.2	28.1
Zen-7B + PRM	76.2	96.8	40.1	77.4	29.8
Zen-32B	78.8	97.2	44.3	81.2	33.4
Zen-32B + PRM	83.4	97.9	48.2	84.7	36.9

Table 3: Mathematical reasoning benchmark results. PRM adds 4–8 points across benchmarks.

Note: The headline Zen-7B result of 73.2% MATH cited in the abstract reflects the production model with best-of-32 sampling; single-sample accuracy is 71.4%.

5.2 MATH Category Breakdown

Category	Zen-7B (%)	Zen-7B + PRM (%)
Algebra	84.3	88.1
Counting & Probability	71.2	76.4
Geometry	62.4	67.8
Number Theory	78.3	82.1
Precalculus	66.1	71.2
Intermediate Algebra	69.8	74.3
Overall	71.4	76.2

Table 4: MATH benchmark by category. Geometry benefits most from PRM guidance.

5.3 Effect of Sampling at Test Time

Samples	1	4	16	32	64
No PRM (majority vote)	71.4	74.1	76.3	77.2	77.8
PRM (best-of- k)	71.4	74.8	77.3	78.1	79.0

Table 5: MATH accuracy vs. number of samples (Zen-7B). PRM reranking outperforms majority vote.

5.4 PRM Calibration

The PRM step correctness scores are well-calibrated: across 50K held-out steps, steps scored < 0.1 have 6.2% actual correctness rate; steps scored > 0.9 have 91.4% actual correctness rate. Calibration error (ECE) = 0.043.

6 Analysis

6.1 Error Taxonomy

We manually analyzed 200 errors by Zen-7B on MATH Level 5 problems:

Error Type	Count	PRM Detectable?
Arithmetic computation error	61	78%
Wrong approach / setup	44	52%
Algebraic manipulation error	38	71%
Missing case / edge case	29	31%
Incorrect interpretation	18	18%
Copy/transcription error	10	90%
Total	200	

Table 6: Error taxonomy for Zen-7B on MATH Level 5. PRM detects most computational errors.

The PRM is most effective at detecting arithmetic and algebraic errors (step produces numerically inconsistent result) and least effective at detecting missing cases or misinterpretation (where each individual step appears locally valid).

6.2 Olympiad-Level Performance

On AIME 2024 (30 problems), Zen-32B with PRM-guided search and $n = 128$ samples achieves 48.2% (14.5/30 problems). Human AIME competitors typically score 8–15 in competitive settings.

On FrontierMath (research-level problems from active mathematics research), 36.9% represents a significant capability frontier. Many FrontierMath problems require theorem-proving capabilities beyond current neural approaches.

7 Conclusion

Mathematical reasoning in Zen models benefits substantially from three components: domain-rich pretraining data (120B tokens), process reward models that provide step-level guidance (+4–8 MATH points), and hybrid verification enabling reliable self-evaluation. The combination yields 83.4% MATH, 97.9% GSM8K, and 48.2% AIME for Zen-32B. Open problems include proof-level verification for advanced mathematics and step-level data collection at Olympiad difficulty.

References

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